

# Constraints in Hamiltonian time-dependent mechanics

GIOVANNI GIACHETTA<sup>1</sup>, LUIGI MANGIAROTTI<sup>1</sup>

AND GENNADI SARDANASHVILY<sup>2</sup>

<sup>1</sup> Department of Mathematics and Physics, Camerino University, 62032 Camerino, Italy

E-mail: mangiaro@camserv.unicam.it

<sup>2</sup> Department of Theoretical Physics, Moscow State University, 117234 Moscow, Russia

E-mail: sard@grav.phys.msu.su

**Abstract** In Hamiltonian time-dependent mechanics, the Poisson bracket does not define dynamic equations, that implies the corresponding peculiarities of describing time-dependent holonomic constraints. As in conservative mechanics, one can consider the Poisson bracket of constraints, separate them in first and second class constraints, construct the Koszul–Tate resolution and a BRST complex. However, the Poisson bracket of constraints and a Hamiltonian makes no sense. Hamiltonian vector fields for first class constraints are not generators of gauge transformations. In the case of Lagrangian constraints, we state the comprehensive relations between solutions of the Lagrange equations for an almost regular Lagrangian and solutions of the Hamilton equations for associated Hamiltonian forms, which live in the Lagrangian constraint space. Degenerate quadratic Lagrangian systems are studied in details. We construct the Koszul–Tate resolution for Lagrangian constraints of these systems in an explicit form.

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## 1 Introduction

The technique of Poisson and symplectic manifolds is well known to provide the adequate Hamiltonian formulation of classical and quantum conservative mechanics. This is also the case of presymplectic Hamiltonian systems. Since every presymplectic form can be represented as a pull-back of a symplectic form by a coisotropic imbedding [14, 23], a presymplectic Hamiltonian system can be seen as a Dirac constraint system [2, 23]. An autonomous Lagrangian system also exemplifies a presymplectic Hamiltonian system where a presymplectic form is the exterior differential of the Poincaré–Cartan form, while a Hamiltonian is the energy function [6, 21, 23, 27]. A generic example of conservative Hamiltonian mechanics is a regular Poisson manifold  $(Z, w)$  where a Hamiltonian is a real function  $\mathcal{H}$  on  $Z$ . Given the corresponding Hamiltonian vector field  $\vartheta_{\mathcal{H}} = w^{\sharp}(d\mathcal{H})$ , the closed subbundle  $\vartheta_{\mathcal{H}}(Z)$  of the tangent bundle  $TZ$  is an autonomous first order dynamic equation on a manifold  $Z$ , called the Hamilton equations. The evolution equation on the Poisson algebra  $C^{\infty}(Z)$  is the Lie derivative  $\mathbf{L}_{\vartheta_{\mathcal{H}}}f = \{\mathcal{H}, f\}$ , expressed into the Poisson bracket of the Hamiltonian  $\mathcal{H}$  and functions  $f$  on  $Z$ . This description, however, cannot be extended in a straightforward manner to time-dependent mechanics subject to time-dependent transformations.

The existent formulations of time-dependent mechanics imply usually a preliminary splitting of a configuration space  $Q = \mathbf{R} \times M$  and a momentum phase space  $\Pi = \mathbf{R} \times Z$ , where  $Z$  is a Poisson manifold [5, 7, 8, 17, 26, 20]. From the physical viewpoint, this means that a certain reference frame is chosen. In this case, the momentum phase space  $\Pi$  is endowed with the Poisson product of the zero Poisson structure on  $\mathbf{R}$  and the Poisson structure on  $Z$ . A Hamiltonian is defined as a real function  $\mathcal{H}$  on  $\Pi$ . The corresponding Hamiltonian vector field  $\vartheta_{\mathcal{H}}$  on  $\Pi$  is vertical

with respect to the fibration  $\Pi \rightarrow \mathbf{R}$ . Due to the canonical imbedding

$$\Pi \times_{\mathbf{R}} T\mathbf{R} \rightarrow T\Pi, \quad (1.1)$$

one introduces the vector field

$$\gamma_{\mathcal{H}} = \partial_t + \vartheta_{\mathcal{H}}, \quad (1.2)$$

where  $\partial_t$  is the standard vector field on  $\mathbf{R}$  [17]. The first order dynamic equation  $\gamma_{\mathcal{H}}(\Pi) \subset T\Pi$  on the manifold  $\Pi$  plays the role of Hamilton equations. The evolution equation on the Poisson algebra  $C^\infty(\Pi)$  is given by the Lie derivative

$$\mathbf{L}_{\gamma_{\mathcal{H}}} f = \partial_t f + \{\mathcal{H}, f\}. \quad (1.3)$$

This is not the case of mechanical systems subject to time-dependent transformations. These transformations, including canonical and inertial frame transformations, violate the splitting  $\mathbf{R} \times Z$ . As a consequence, there is no canonical imbedding (1.1), and the vector field (1.2) is not well defined. At the same time, one can treat the imbedding (1.1) as a trivial connection on the bundle  $\Pi \rightarrow \mathbf{R}$ , while  $\gamma_{\mathcal{H}}$  (1.2) is the sum of the horizontal lift onto  $\Pi$  of the vector field  $\partial_t$  by this connection and of the vertical vector field  $\vartheta_{\mathcal{H}}$ . This observation make us to think of non-relativistic time-dependent mechanics as being a particular field theory on fibre bundles over  $\mathbf{R}$ , where the time axis  $\mathbf{R}$  is parameterized by the Cartesian coordinates  $t$  with the transition functions  $t' = t + \text{const}$ . Then  $\mathbf{R}$  is provided with the above mentioned standard vector field  $\partial_t$  and the standard 1-form  $dt$ . Every fibre bundle over  $\mathbf{R}$  is obviously trivial, but its trivialization is not necessarily canonical.

*Remark 1.1.* The following peculiarity of bundles over  $\mathbf{R}$  is important. Let  $Y \rightarrow \mathbf{R}$  be a fibre bundle coordinated by  $(t, y^A)$ , and  $J^1 Y$  its first order jet manifold, equipped with the adapted coordinates  $(t, y^A, y_t^A)$ . There is the canonical imbedding

$$\lambda = \partial_t + y_t^A \partial_A : J^1 Y \xrightarrow{Y} TY \quad (1.4)$$

onto the affine subbundle of  $TY \rightarrow Y$  of elements  $v \in TY$  such that  $v \rfloor dt = 1$ . This subbundle is modelled over the vertical tangent bundle  $VY \rightarrow Y$ . As a consequence, there is one-to-one correspondence between the connections  $\Gamma$  on the fibre bundle  $Y \rightarrow \mathbf{R}$ , treated as sections of the affine jet bundle  $\pi_0^1 : J^1 Y \rightarrow Y$  [24], and the

nowhere vanishing vector fields  $\Gamma = \partial_t + \Gamma^A \partial_A$  on  $Y$ , called horizontal vector fields, such that  $\Gamma]dt = 1$  [23, 24]. The corresponding covariant differential reads

$$D_\Gamma = \lambda - \Gamma : J^1Y \xrightarrow{Y} VY, \quad \dot{y}^A \circ D_\Gamma = y_t^A - \Gamma^A.$$

Let us also recall the total derivative  $d_t = \partial_t + y_t^A \partial_A + \dots$  and the exterior algebra homomorphism

$$h_0 : \phi dt + \phi_A dy^A \mapsto (\phi + \phi_A y_t^A) dt \quad (1.5)$$

which sends exterior forms on  $Y \rightarrow \mathbf{R}$  onto the horizontal forms on  $J^1Y \rightarrow \mathbf{R}$ , and vanishes on contact forms  $\theta^A = dy^A - y_t^A dt$ .

Lagrangian time-dependent mechanics follows directly Lagrangian field theory [11, 19, 22, 23, 25]. It implies the existence of a configuration space  $Q \rightarrow \mathbf{R}$  of a mechanical system, and a Lagrangian is defined as a horizontal density

$$L = \mathcal{L} dt, \quad \mathcal{L} : J^1Q \rightarrow \mathbf{R}, \quad (1.6)$$

on the velocity phase space  $J^1Q$ . However, there is the essential difference between field theory and time-dependent mechanics. The curvature of any connection  $\Gamma$  on a configuration bundle  $Q \rightarrow \mathbf{R}$  vanishes identically, and these connections fail to be dynamic variables, but characterize reference frames. The horizontal vector field  $\Gamma$  sets a tangent vector at each point of the configuration space  $Q$ , which can be seen as the velocity of an "observer" at this point [23, 25, 31]. There is the correspondence between the connections  $\Gamma$  on the configuration bundle  $Q \rightarrow \mathbf{R}$  and the trivializations of  $Q \rightarrow \mathbf{R}$  such that  $\Gamma = \partial_t$  in the adapted coordinates (see Section 2).

A generic momentum phase space of time-dependent mechanics is a fibre bundle  $\Pi \rightarrow \mathbf{R}$  endowed with a regular Poisson structure whose characteristic distribution belongs to the vertical tangent bundle  $V\Pi$  of  $\Pi \rightarrow \mathbf{R}$  [17]. Such a Poisson structure however cannot provide dynamic equations. A first order dynamic equation on  $\Pi \rightarrow \mathbf{R}$ , by definition, is a section of the affine jet bundle  $J^1\Pi \rightarrow \Pi$ , i.e., a connection on  $\Pi \rightarrow \mathbf{R}$ . Being a horizontal vector field, such a connection cannot be a Hamiltonian vector field with respect to the above mentioned Poisson structure on  $\Pi$ .

One can overcome this difficulty as follows. Let  $Q \rightarrow \mathbf{R}$  be a configuration bundle of time-dependent mechanics. The corresponding momentum phase space is the vertical cotangent bundle  $\Pi = V^*Q \rightarrow \mathbf{R}$ , called the Legendre bundle, while

the cotangent bundle  $T^*Q$  is the homogeneous momentum phase space.  $T^*Q$  admits the canonical Liouville form  $\Xi$  and the symplectic form  $d\Xi$ , together with the corresponding non-degenerate Poisson bracket  $\{, \}_T$  on the ring  $C^\infty(T^*Q)$ . Let us consider the subring of  $C^\infty(T^*Q)$  which comprises the pull-backs  $\zeta^*f$  onto  $T^*Q$  of functions  $f$  on the vertical cotangent bundle  $V^*Q$  by the canonical fibration

$$\zeta : T^*Q \rightarrow V^*Q. \quad (1.7)$$

This subring is closed under the Poisson bracket  $\{, \}_T$ , and  $V^*Q$  is provided with the regular Poisson structure  $\{, \}_V$  such that

$$\zeta^*\{f, g\}_V = \{\zeta^*f, \zeta^*g\}_T \quad (1.8)$$

[32]. Its characteristic distribution coincides with the vertical tangent bundle  $VV^*Q$  of  $V^*Q \rightarrow \mathbf{R}$ . Given a section  $h$  of the bundle (1.7), let us consider the pull-back forms

$$\Theta = h^*(\Xi \wedge dt), \quad \Omega = h^*(d\Xi \wedge dt) \quad (1.9)$$

on  $V^*Q$ , but these forms are independent of a section  $h$  and are canonical exterior forms on  $V^*Q$ . The pull-backs  $h^*\Xi$  are called the Hamiltonian forms. With  $\Omega$ , the Hamiltonian vector field  $\vartheta_f$  for a function  $f$  on  $V^*Q$  is given by the relation

$$\vartheta_f \lrcorner \Omega = -df \wedge dt, \quad (1.10)$$

while the Poisson bracket (1.8) is written as

$$\{f, g\}_V dt = \vartheta_g \lrcorner \vartheta_f \lrcorner \Omega.$$

Note that a generic momentum phase space  $\Pi \rightarrow \mathbf{R}$  of time-dependent mechanics can be seen locally as the Poisson product over  $\mathbf{R}$  of the Legendre bundle  $V^*Q \rightarrow \mathbf{R}$  and a fibre bundle over  $\mathbf{R}$ , equipped with the zero Poisson structure.

The pair  $(V^*Q, \Omega)$  is the particular ( $n = 1$ )-polysymplectic phase space of the covariant Hamiltonian field theory (see [4, 12, 15, 29] for a survey). Following its general scheme, we can formulate the Hamiltonian time-dependent mechanics as follows [23, 31].

A connection  $\gamma$  on the Legendre bundle  $V^*Q \rightarrow \mathbf{R}$  is called canonical if the corresponding horizontal vector field is canonical for the Poisson structure on  $V^*Q$ ,

i.e., the form  $\gamma \rfloor \Omega$  is closed. We will prove that such a form is necessarily exact. A canonical connection  $\gamma$  is said to be a Hamiltonian connection if

$$\gamma \rfloor \Omega = dH \quad (1.11)$$

where  $H$  is a Hamiltonian form on  $V^*Q$ . We show that every Hamiltonian form admits a unique Hamiltonian connection  $\gamma_H$ , and that any canonical connection is locally a Hamiltonian one. Given a Hamiltonian form  $H$ , the kernel of the covariant differential  $D_{\gamma_H}$ , associated with the Hamiltonian connection  $\gamma_H$ , is a closed imbedded subbundle of the jet bundle  $J^1 V^*Q \rightarrow \mathbf{R}$ , and so is the system of first order PDEs on the Legendre bundle  $V^*Q \rightarrow \mathbf{R}$ . These are the Hamilton equations in time-dependent mechanics, while the Lie derivative

$$\mathbf{L}_{\gamma_H} f = \gamma_H \rfloor df \quad (1.12)$$

defines the evolution equation on  $C^\infty(V^*Z)$ . As in the polysymplectic case [12, 28, 29], this Hamiltonian dynamics is equivalent to the Lagrangian one for hyperregular Lagrangians, while a degenerate Lagrangian involves a set of associated Hamiltonian forms in order to exhaust solutions of the Lagrange equations.

The main peculiarity of Hamiltonian time-dependent mechanics lies in the fact that, since  $\gamma_H$  is not a vertical vector field, the right-hand side of the evolution equation (1.12) is not expressed into the Poisson bracket in a canonical way, but contains a frame-dependent term. Every connection  $\Gamma$  on the configuration bundle  $Q \rightarrow \mathbf{R}$  is an affine section of the bundle (1.7), and defines the Hamiltonian form  $H_\Gamma = \Gamma^* \Xi$  on  $V^*Q$ . The corresponding Hamiltonian connection is the canonical lift  $V^*\Gamma$  of  $\Gamma$  onto the Legendre bundle  $V^*Q$  [12, 24]. Then any Hamiltonian form  $H$  on  $V^*Q$  admits splittings

$$H = H_\Gamma - \widetilde{\mathcal{H}}_\Gamma dt, \quad (1.13)$$

$$\gamma_H = V^*\Gamma + \vartheta_{\widetilde{\mathcal{H}}_\Gamma},$$

where  $\vartheta_{\widetilde{\mathcal{H}}_\Gamma}$  is the vertical Hamiltonian field for the function  $\widetilde{\mathcal{H}}_\Gamma$ , which the energy function with respect to the reference frame  $\Gamma$  (see Section 4). With the splitting (1.13), the evolution equation (1.12) takes the form

$$\mathbf{L}_{\gamma_H} f = V^*\Gamma \rfloor H + \{\widetilde{\mathcal{H}}_\Gamma, f\}_V. \quad (1.14)$$

Let the configuration bundle  $Q \rightarrow \mathbf{R}$  with an  $m$ -dimensional typical fibre  $M$  be coordinated by  $(t, q^i)$ . Then Legendre bundle  $V^*Q$  and the cotangent bundle  $T^*Q$

are provided with holonomic coordinates  $(t, q^i, p_i = \dot{q}_i)$  and  $(t, q^i, p_i, p)$ , respectively. Relative to these coordinates, a Hamiltonian form  $H$  on  $V^*Q$  reads

$$H = h^*\Xi = p_i dq^i - \mathcal{H}dt. \quad (1.15)$$

It is the well-known integral invariant of Poincaré–Cartan, where  $\mathcal{H}$  is a Hamiltonian in time-dependent mechanics. A glance at the expression (1.15) shows that  $\mathcal{H}$  fails to be a scalar under time-dependent transformations. Accordingly, the evolution equation (1.14) takes the local form

$$\mathbf{L}_{\gamma_H} = \partial_t f + \{\mathcal{H}, f\}_V, \quad (1.16)$$

but one should bear in mind that the terms in its right-hand side, taken separately, are not well-behaved objects under time-dependent transformations. In particular, the equality  $\{\mathcal{H}, f\}_V = 0$  is not preserved under time-dependent transformations.

The above peculiarities of Hamiltonian time-dependent mechanics imply the corresponding peculiarities of describing time-dependent holonomic constraints. As in conservative mechanics, one can consider the Poisson bracket of constraints, separate them in first and second class constraints, construct the Koszul–Tate resolution and BRST complex. However, the Poisson bracket of constraints and a Hamiltonian makes no sense in time-dependent mechanics. Hamiltonian vector fields for first class constraint functions are not generators of gauge transformations. We will pay a special attention to Lagrangian constraints. Every Lagrangian  $L$  defines the Legendre map

$$\widehat{L} : J^1Q \xrightarrow{Q} V^*Q, \quad p_i \circ \widehat{L} = \pi_i, \quad (1.17)$$

whose image  $N_L = \widehat{L}(J^1Q) \subset V^*Q$  is called the Lagrangian constraint space. We state the comprehensive relationship between solutions of the Lagrange equations for an almost regular Lagrangian  $L$  and solutions in  $N_L$  of the Hamilton equations for associated Hamiltonian forms. The detailed analysis of degenerate quadratic Lagrangian systems in Section 7 is appropriate for application to many physical models. In Section 9, we construct the Koszul–Tate resolution for Lagrangian constraints of such a degenerate system in an explicit form.

## 2 Interlude I. Non-relativistic reference frames

As was mentioned above, a reference frame in non-relativistic mechanics is identified with a connection  $\Gamma$  on the configuration bundle  $Q \rightarrow \mathbf{R}$ . Being flat, every connec-

tion  $\Gamma$  on  $Q \rightarrow \mathbf{R}$  yields an integrable horizontal distribution on  $Q$ , whose integral manifolds are integral curves of the horizontal vector field  $\Gamma$  which are transversal to the fibres of the bundle  $Q \rightarrow \mathbf{R}$ .

**Proposition 2.1.** [12, 24]. *Each connection  $\Gamma$  on a bundle  $Q \rightarrow \mathbf{R}$  defines an atlas of local constant trivializations of  $Q \rightarrow \mathbf{R}$  such that the associated bundle coordinates  $(t, \bar{q}^i)$  on  $Q$  possess the transition function  $\bar{q}^i \rightarrow \bar{q}^i(\bar{q}^j)$  independent of  $t$ , and  $\Gamma = \partial_t$  with respect to these coordinates. Conversely, every atlas of local constant trivializations of the bundle  $Q \rightarrow \mathbf{R}$  sets a connection on  $Q \rightarrow \mathbf{R}$  which is  $\partial_t$  relative to this atlas.*

**Proposition 2.2.** [23, 31]. *Every bundle trivialization*

$$\psi : Q \cong \mathbf{R} \times M \tag{2.1}$$

*yields a complete horizontal vector field  $\Gamma$  on this bundle. Conversely, every complete connection  $\Gamma$  on  $Q \rightarrow \mathbf{R}$  defines its trivialization (2.1) such that  $\Gamma = \partial_t$ .*

One can think of the atlas of local constant trivializations and the bundle coordinates  $(t, \bar{q}^i)$  in Proposition 2.1 as being also a reference frame corresponding to the connection  $\Gamma$ . These coordinates are said to be adapted to the reference frame  $\Gamma$ . In particular, the Hamiltonian form  $H_\Gamma$  relative to the adapted coordinates reduces to the pure kinematic term  $H_\Gamma = p_i dy^i$ . Therefore, we will call  $H_\Gamma$  a frame Hamiltonian form. Unless otherwise stated, by a reference frame will be meant a complete reference frame. Given a trivialization (2.1) of the configuration bundle  $Q \rightarrow \mathbf{R}$ , we have the corresponding trivializations of velocity and momentum phase spaces

$$J^1Q \cong \mathbf{R} \times TM, \quad V^*Q \cong \mathbf{R} \times T^*M.$$

### 3 Interlude II. Lagrangian time-dependent dynamics

To obtain a complete picture of the relations between Lagrangian and Hamiltonian time-dependent mechanics in Section 6, we will refer to the following three types of PDEs in the first order calculus of variations. These are Lagrange, Cartan and Hamilton–De Donder equations.

Given a Lagrangian  $L$  on the velocity phase space  $J^1Q$ , we follow the first variational formula of the calculus of variations [12, 23, 30], which provides the canonical



decomposition of the Lie derivative  $\mathbf{L}_{J^1u}L = (J^1u]\mathcal{L})dt$  of  $L$  along a projectable vector field  $u$  on  $Q \rightarrow \mathbf{R}$ . We have

$$J^1u]\mathcal{L} = u_V]\mathcal{E}_L + d_t(u]H_L), \quad (3.1)$$

where  $u_V = (u]\theta^i)\partial_i$ ,

$$H_L = L + \pi_i\theta^i, \quad \pi_i = \partial_i^t\mathcal{L}, \quad (3.2)$$

is the Poincaré–Cartan form and

$$\mathcal{E}_L = (\partial_i - d_t\pi_i)\mathcal{L}\bar{d}q^i : J^2Q \rightarrow V^*Q \quad (3.3)$$

is the Euler–Lagrange operator associated with  $L$ . The kernel  $\text{Ker } \mathcal{E}_L \subset J^2Q$  of  $\mathcal{E}_L$  defines the Lagrange equations on  $Q$ , given by the coordinate relations

$$(\partial_i - d_t\pi_i)\mathcal{L} = 0. \quad (3.4)$$

On-shell, the first variational formula (3.1) leads to the weak identity

$$\mathbf{L}_{J^1u}L \approx d_t(u]H_L)dt,$$

and then, if  $\mathbf{L}_{J^1u}L = 0$ , to the weak conservation law

$$0 \approx dt(u]H_L) = -d_t\mathcal{T} \quad (3.5)$$

of the symmetry current

$$\mathcal{T} = -(u]H_L) = -\pi_i(u^tq_t^i - u^i) - u^t\mathcal{L}. \quad (3.6)$$

Being the Lepagean equivalent of the Lagrangian  $L$  on  $J^1Q$  (i.e.,  $L = h_0(H_L)$  where  $h_0$  is the morphism (1.5)), the Poincaré–Cartan form  $H_L$  (3.2) is also the Lepagean equivalent of the Lagrangian

$$\bar{L} = \hat{h}_0(H_L) = (\mathcal{L} + (\hat{q}_t^i - q_t^i)\pi_i)dt, \quad \hat{h}_0(dy^i) = \hat{y}_t^i dt, \quad (3.7)$$

on the repeated jet manifold  $J^1J^1Q$ , coordinated by  $(t, q^i, q_t^i, \hat{q}_t^i, q_{tt}^i)$ . The Euler–Lagrange operator  $\mathcal{E}_{\bar{L}} : J^1J^1Q \rightarrow V^*J^1Q$  for  $\bar{L}$  reads

$$\begin{aligned} \mathcal{E}_{\bar{L}} &= (\partial_i\mathcal{L} - \hat{d}_t\pi_i + \partial_i\pi_j(\hat{q}_t^j - q_t^j))\bar{d}q^i + \partial_i^t\pi_j(\hat{q}_t^j - q_t^j)\bar{d}q_t^i, \\ \hat{d}_t &= \partial_t + \hat{q}_t^i\partial_i + q_{tt}^i\partial_i^t. \end{aligned} \quad (3.8)$$

Its kernel  $\text{Ker } \mathcal{E}_{\bar{L}} \subset J^1 J^1 Y$  defines the Cartan equations

$$\partial_i^t \pi_j (\hat{q}_t^j - q_t^j) = 0, \quad \partial_i \mathcal{L} - \hat{d}_t \pi_i + (\hat{q}_t^j - q_t^j) \partial_i \pi_j = 0. \quad (3.9)$$

Since  $\mathcal{E}_{\bar{L}}|_{J^2 Q} = \mathcal{E}_L$ , the Cartan equations (3.9) are equivalent to the Lagrange equations (3.4) on integrable sections  $\bar{c} = \dot{c}$  of  $J^1 Q \rightarrow \mathbf{R}$ . These equations are equivalent in the case of regular Lagrangians.

On sections  $\bar{c} : \mathbf{R} \rightarrow J^1 Q$ , the Cartan equations (3.9) are equivalent to the relation

$$\bar{c}^*(u \rfloor dH_L) = 0 \quad (3.10)$$

which is assumed to hold for all vertical vector fields  $u$  on  $J^1 Q \rightarrow \mathbf{R}$ .

With the Poincaré–Cartan form  $H_L$  (3.2), we have the Legendre morphism

$$\widehat{H}_L : J^1 Q \rightarrow T^*Q, \quad (p_i, p) \circ \widehat{H}_L = (\pi_i, \mathcal{L} - \pi_i q_t^i).$$

Let  $Z_L = \widehat{H}_L(J^1 Q)$  be an imbedded subbundle  $i_L : Z_L \hookrightarrow T^*Q$  of  $T^*Q \rightarrow Q$ . It is provided with the pull-back De Donder form  $i_L^* \Xi$ . We have

$$H_L = \widehat{H}_L^* \Xi_L = \widehat{H}_L^* (i_L^* \Xi). \quad (3.11)$$

By analogy with the Cartan equations (3.10), the Hamilton–De Donder equations for sections  $\bar{r}$  of  $T^*Q \rightarrow \mathbf{R}$  are written as

$$\bar{r}^*(u \rfloor d\Xi_L) = 0 \quad (3.12)$$

where  $u$  is an arbitrary vertical vector field on  $T^*Q \rightarrow \mathbf{R}$ .

**Theorem 3.1.** [15]. *Let the Legendre morphism  $\widehat{H}_L : J^1 Q \rightarrow Z_L$  be a submersion. Then a section  $\bar{c}$  of  $J^1 Q \rightarrow \mathbf{R}$  is a solution of the Cartan equations (3.10) iff  $\widehat{H}_L \circ \bar{c}$  is a solution of the Hamilton–De Donder equations (3.12), i.e., Cartan and Hamilton–De Donder equations are quasi-equivalent.*

## 4 Hamiltonian time-dependent dynamics

Let the Legendre bundle  $V^*Q \rightarrow \mathbf{R}$  be provided with the holonomic coordinates  $(t, q^i, q_t^i)$ . Relative to these coordinates, the canonical 3-form  $\Omega$  (1.9) and the canonical Poisson structure (1.8) on  $V^*Q$  read

$$\Omega = dp_i \wedge dq^i \wedge dt, \quad (4.1)$$

$$\{f, g\}_V = \partial^i f \partial_i g - \partial^i g \partial_i f, \quad f, g \in C^\infty V^*Q. \quad (4.2)$$

The corresponding symplectic foliation coincides with the fibration  $V^*Q \rightarrow \mathbf{R}$ . The symplectic forms on the fibres of  $V^*Q \rightarrow \mathbf{R}$  are the pull-backs  $\Omega_t = dp_i \wedge dq^i$  of the canonical symplectic form on the typical fibre  $T^*M$  of the Legendre bundle  $V^*Q \rightarrow \mathbf{R}$  with respect to trivialization morphisms [3, 17, 31]. Given such a trivialization, the Poisson structure (4.2) is isomorphic to the product of the zero Poisson structure on  $\mathbf{R}$  and the canonical symplectic structure on  $T^*M$ .

*Remark 4.1.* It is easily seen that an automorphism  $\rho$  of the Legendre bundle  $V^*Q \rightarrow \mathbf{R}$  is a canonical transformation of the Poisson structure (4.2) iff it preserves the canonical 3-form  $\Omega$  (4.1). Let us emphasize that canonical transformations are compatible with the fibration  $V^*Q \rightarrow \mathbf{R}$ , but not necessarily with the fibration  $\pi_Q : V^*Q \rightarrow Q$ . We will restrict ourselves to the holonomic coordinates on  $V^*Y$  and holonomic transformations which are obviously canonical.

With respect to the Poisson bracket (4.2), the Hamiltonian vector field  $\vartheta_f$  for a function  $f$  on the momentum phase space  $V^*Q$  is

$$\vartheta_f = \partial^i f \partial_i - \partial_i f \partial^i. \quad (4.3)$$

A Hamiltonian vector field, by definition, is canonical. A converse is the following.

**Proposition 4.1.** *Every vertical canonical vector field on the Legendre bundle  $V^*Q \rightarrow \mathbf{R}$  is locally a Hamiltonian vector field.*

The proof is based on the following facts.

**Lemma 4.2.** *Let  $\sigma$  be a 1-form on  $V^*Q$ . If  $\sigma \wedge dt$  is closed form, it is exact.*

*Proof.* Since  $V^*Q$  is diffeomorphic to  $\mathbf{R} \times T^*M$ , we have the De Rham cohomology group

$$H^2(V^*Q) = H^0(\mathbf{R}) \otimes H^2(T^*M) \oplus H^1(\mathbf{R}) \otimes H^1(T^*M).$$

The form  $\sigma \wedge dt$  belongs to its second item which is zero.  $\square$

**Lemma 4.3.** *If the 2-form  $\sigma \wedge dt$  is exact, then  $\sigma \wedge dt = dg \wedge dt$  locally.*

*Proof.* The proof is based on the relative Poincaré lemma [12].  $\square$

Let  $\gamma = \partial_t + \gamma^i \partial_i + \gamma_i \partial^i$  be a canonical connection on the Legendre bundle  $V^*Q \rightarrow \mathbf{R}$ . Its components obey the relations

$$\partial^i \gamma^j - \partial^j \gamma^i = 0, \quad \partial_i \gamma_j - \partial_j \gamma_i = 0, \quad \partial_j \gamma^i + \partial^i \gamma_j = 0. \quad (4.4)$$

Canonical connections constitute an affine space modelled over the vector space of vertical canonical vector fields on  $V^*Q \rightarrow \mathbf{R}$ .

**Proposition 4.4.** *If  $\gamma$  is a canonical connection, then the form  $\gamma \rfloor \Omega$  is exact.*

*Proof.* Every connection  $\Gamma$  on  $Q \rightarrow \mathbf{R}$  gives rise to the connection

$$V^*\Gamma = \partial_t + \Gamma^i \partial_i - p_i \partial_j \Gamma^i \partial^j \quad (4.5)$$

on  $V^*Q \rightarrow \mathbf{R}$  which is a Hamiltonian connection for the frame Hamiltonian form

$$V^*\Gamma \rfloor \Omega = dH_\Gamma, \quad H_\Gamma = p_i dq^i - p_i \Gamma^i dt. \quad (4.6)$$

Let us consider the decomposition  $\gamma = V^*\Gamma + \vartheta$ , where  $\Gamma$  is a connection on  $Q \rightarrow \mathbf{R}$ . The assertion follows from the relation (4.6) and Proposition 4.1.  $\square$

Thus, every canonical connection  $\gamma$  on  $V^*Q$  defines an exterior 1-form  $H$  modulo closed forms so that  $dH = \gamma \rfloor \Omega$ . Such a form is called a locally Hamiltonian form.

**Proposition 4.5.** *Every locally Hamiltonian form on the momentum phase space  $V^*Q$  is locally a Hamiltonian form modulo closed forms.*

*Proof.* Given locally Hamiltonian forms  $H_\gamma$  and  $H_{\gamma'}$ , their difference  $\sigma = H_\gamma - H_{\gamma'}$  is a 1-form on  $V^*Q$  such that the 2-form  $\sigma \wedge dt$  is closed. By virtue of Lemmas 4.2 and 4.3, the form  $\sigma \wedge dt$  is exact and  $\sigma = f dt + dg$  locally. Put  $H_{\gamma'} = H_\Gamma$  where  $\Gamma$  is a connection on  $V^*Q \rightarrow \mathbf{R}$ . Then  $H_\gamma$  modulo closed forms takes the local form  $H_\gamma = H_\Gamma + f dt$ , and coincides with the pull-back of the Liouville form  $\Xi$  on  $T^*Q$  by the local section  $p = -p_i \Gamma^i + f$  of the fibre bundle (1.7).  $\square$

**Proposition 4.6.** *Conversely, each Hamiltonian form  $H$  on the momentum phase space  $V^*Q$  admits a unique canonical connection  $\gamma_H$  on  $V^*Q \rightarrow \mathbf{R}$  such that the relation (1.11) holds.*

*Proof.* Given a Hamiltonian form  $H$ , its exterior differential

$$dH = h^* d\Xi = (dp_i + \partial_i \mathcal{H} dt) \wedge (dq^i - \partial^i \mathcal{H} dt) \quad (4.7)$$

is a presymplectic form of constant rank  $2m$  since the form

$$(dH)^m = (dp_i \wedge dq^i)^m - m(dp_i \wedge dq^i)^{m-1} \wedge d\mathcal{H} \wedge dt \quad (4.8)$$

is nowhere vanishing. It is also seen that  $(dH)^m \wedge dt \neq 0$ . It follows that the kernel of  $dH$  is a 1-dimensional distribution. Then the desired Hamiltonian connection

$$\gamma_H = \partial_t + \partial^i \mathcal{H} \partial_i - \partial_i \mathcal{H} \partial^i \quad (4.9)$$

is a unique vector field  $\gamma_H$  on  $V^*Q$  such that  $\gamma_H \rfloor dH = 0$ ,  $\gamma_H \rfloor dt = 1$ .  $\square$

*Remark 4.2.* Hamiltonian forms constitute an affine space modelled over the vector space of horizontal densities  $f dt$  on  $V^*Q \rightarrow \mathbf{R}$ , i.e., over  $C^\infty(V^*Q)$ . Accordingly Hamiltonian connections  $\gamma_H$  form an affine space modelled over the vector space of Hamiltonian vector fields. Every Hamiltonian form  $H$  defines the associated Hamiltonian map

$$\widehat{H} = J^1 \pi_Q \circ \gamma_H : \partial_t + \partial^i \mathcal{H} : V^*Q \rightarrow J^1 Q. \quad (4.10)$$

With the Hamiltonian map (4.10), we have another Hamiltonian form

$$H_{\widehat{H}} = -\widehat{H} \rfloor \Theta = p_i dq^i - p_i \partial^i \mathcal{H}. \quad (4.11)$$

It is readily observed that  $H_{\widehat{H}} = H$  iff  $H$  is a frame Hamiltonian form.

Given a Hamiltonian connection  $\gamma_H$  (4.9), the corresponding Hamilton equations  $D_{\gamma_H} = 0$  take the coordinate form

$$q_t^i = \partial^i \mathcal{H}, \quad (4.12a)$$

$$p_{ti} = -\partial_i \mathcal{H}. \quad (4.12b)$$

Their classical solutions are integral sections of the Hamiltonian connection  $\gamma_H$ , i.e.,  $\dot{r} = \gamma_H \circ r$ . On sections  $r$  of the Legendre bundle  $V^*Q \rightarrow \mathbf{R}$ , the Hamilton equations (4.12a) – (4.12b) are equivalent to the relation

$$r^*(u \rfloor dH) = 0 \quad (4.13)$$

which is assumed to hold for any vertical vector field  $u$  on  $V^*Q \rightarrow \mathbf{R}$ .

The following two constructions are useful.

It is readily observed that a Hamiltonian form  $H$  (1.15) is the Poincaré–Cartan form (3.2) for the Lagrangian

$$L_H = h_0(H) = (p_i q_t^i - \mathcal{H})\omega \quad (4.14)$$

on the jet manifold  $J^1V^*Q$ . Given a projectable vector field  $u$  on the configuration bundle  $Q \rightarrow \mathbf{R}$  and its lift

$$\tilde{u} = u^t \partial_t + u^i \partial_i - \partial_i u^j p_j \partial^i \quad (4.15)$$

onto the Legendre bundle  $V^*Q \rightarrow \mathbf{R}$ , we have

$$\mathbf{L}_{\tilde{u}}H = \mathbf{L}_{J^1\tilde{u}}L_H. \quad (4.16)$$

It is easily seen that the Hamilton equations (4.12a) – (4.12b) for  $H$  are exactly the Lagrange equations for  $L_H$ , i.e., they characterize the kernel of the Euler–Lagrange operator

$$\mathcal{E}_H = (q_t^i - \partial^i \mathcal{H}) \bar{d}p_i - (p_{ti} + \partial_i \mathcal{H}) \bar{d}q^i : J^1V^*Q \rightarrow V^*V^*Q \quad (4.17)$$

for the Lagrangian  $L_H$ , called the Hamilton operator for  $H$ .

Using the relation (4.16), let us obtain the Hamiltonian conservation laws in time-dependent mechanics. As in field theory, by gauge transformations in time-dependent mechanics are meant automorphism of the configuration bundle  $Q \rightarrow \mathbf{R}$ , but only over translations of the base  $\mathbf{R}$ . Then, projectable vector fields

$$u = u^t \partial_t + u^i \partial_i, \quad u \rfloor dt = u^t = \text{const.}, \quad (4.18)$$

on  $V^*Q \rightarrow \mathbf{R}$  can be seen as generators of local 1-parameter groups of local gauge transformations. Given a Hamiltonian form  $H$  (1.15), its Lie derivative (4.16) reads

$$\mathbf{L}_{\tilde{u}}H = \mathbf{L}_{J^1\tilde{u}}L_H = (-u^t \partial_t \mathcal{H} + p_i \partial_t u^i - u^i \partial_i \mathcal{H} + \partial_j u^i p_i \partial^j \mathcal{H}) dt. \quad (4.19)$$

The first variational formula (3.1) applied to the Lagrangian  $L_H$  (4.14) leads to the weak identity  $\mathbf{L}_{\tilde{u}}H \approx d_t(u \rfloor H) dt$ . If the Lie derivative (4.19) vanishes, we have the conserved symmetry current

$$J_u = u \rfloor dH = p_i u^i - u^t \mathcal{H}, \quad (4.20)$$

along  $u$ . Every vector field (4.18) is a superposition of a vertical vector field and a reference frame on  $Q \rightarrow \mathbf{R}$ . If  $u$  is a vertical vector field,  $J_u$  is the Nöther current

$$J_u(q) = u \rfloor q = p_i u^i, \quad q = p_i \bar{d}q^i \in V^*Q. \quad (4.21)$$

The symmetry current along a reference frame  $\Gamma$

$$J_\Gamma = p_i \Gamma^i - \mathcal{H} = -\widetilde{\mathcal{H}}_\Gamma \quad (4.22)$$

is the energy function with respect to the reference frame  $\Gamma$ , taken with the sign minus [9, 23, 31]. It is readily observed that, given a Hamiltonian form  $H$ , the energy functions  $\widetilde{\mathcal{H}}_\Gamma$  constitute an affine space modelled over the vector space of Nöther currents.

**Proposition 4.7.** *Given a Hamiltonian form  $H$ , the conserved currents (4.20) form a Lie algebra with respect to the Poisson bracket*

$$\{J_u, J_{u'}\}_V = J_{[u, u']}. \quad (4.23)$$

The second of the above mentioned constructions enables us to represent the right-hand side of the evolution equation (1.16) as a pure Poisson bracket. Given a Hamiltonian form  $H = h^*\Xi$ , let us consider its pull-back  $\zeta^*H$  onto the cotangent bundle  $T^*Q$ . It is readily observed that the difference  $\Xi - \zeta^*H$  is a horizontal 1-form on  $T^*Q \rightarrow \mathbf{R}$ , while

$$\mathcal{H}^* = \partial_t \lrcorner (\Xi - \zeta^*H) = p + \mathcal{H} \quad (4.24)$$

is a function on  $T^*Q$ . Then the relation

$$\zeta^*(\mathbf{L}_{\gamma_H} f) = \{\mathcal{H}^*, \zeta^* f\}_T \quad (4.25)$$

holds for every function  $f \in C^\infty(V^*Q)$ . In particular, given a projectable vector field  $u$  (4.18), the symmetry current  $J_u$  (4.20) is conserved if and only if

$$\{\mathcal{H}^*, \zeta^* J_u\}_T = 0. \quad (4.26)$$

Moreover, let  $\vartheta_{\mathcal{H}^*}$  be the Hamiltonian vector field for the function  $\mathcal{H}^*$  (4.24) with respect to the canonical Poisson structure  $\{, \}_T$  on  $T^*Q$ . Then

$$T\zeta(\vartheta_{\mathcal{H}^*}) = \gamma_H. \quad (4.27)$$

## 5 Time-dependent constraints

As was mentioned above, an algebra of time-dependent constraints on the momentum phase space  $V^*Q$  can be described similarly to that in conservative Hamiltonian mechanics.

Let  $N$  be a closed imbedded subbundle  $i_N : N \hookrightarrow V^*Q$  of the Legendre bundle  $V^*Q \rightarrow \mathbf{R}$ , treated as a constraint space. Let us consider the ideal  $I_N$  of real

functions  $f$  on  $V^*Q$  which vanish on  $N$ , i.e.,  $i_N^*f = 0$ . Its elements are said to be constraints. There is the isomorphism

$$C^\infty(V^*Q)/I_N \cong C^\infty(N) \quad (5.1)$$

of associative commutative algebras.  $N$  cannot be neither Lagrangian nor symplectic submanifold with respect to the Poisson structure on  $V^*Q$ . By the normalize  $\bar{I}_N$  of the ideal  $I_N$  is meant the subset of functions of  $C^\infty(V^*Q)$  whose Hamiltonian vector fields restrict to vector fields on  $N$ , i.e.,

$$\bar{I}_N = \{f \in C^\infty(V^*Q) : \{f, g\}_V \in I_N, \forall g \in I_N\}, \quad (5.2)$$

[18]. It follows from the Jacobi identity that the normalizer (5.2) is a Poisson subalgebra of  $C^\infty(V^*Q)$ . Put

$$I'_N = \bar{I}_N \cap I_N. \quad (5.3)$$

It is naturally a Poisson subalgebra of  $\bar{I}_N$ . Its elements are called the first class constraints, while the remaining elements of  $I_N$  are the second class constraints. It is readily observed that  $I_N^2 \subset I'_N$ , i.e., the products of second class constraints are first class constraints.

*Remark 5.1.* Let  $N$  be a coisotropic submanifold of  $V^*Q$ , i.e.,  $w^\#(\text{Ann } TN) \subset TN$ . Then  $I_N \subset \bar{I}_N$  and  $I_N = I'_N$ , i.e., all constraints are of the first class.

The relation (4.25) enables us to extend the constraint algorithm of conservative mechanics and time-dependent mechanics on a product  $\mathbf{R} \times M$  (see [7, 20]) to mechanical systems subject to time-dependent transformations.

Let  $H$  be a Hamiltonian form on the momentum phase space  $V^*Q$ . In accordance with the relation (4.25), a constraint  $f \in I_N$  is preserved if the bracket (4.25) vanishes. It follows that the solutions of the Hamilton equations (4.12a) – (4.12b) do not leave the constraint space  $N$  if

$$\{\mathcal{H}^*, \zeta^* I_N\}_T \subset \zeta^* I_N. \quad (5.4)$$

If the relation (5.4) fails to hold, let us introduce secondary constraints  $\{\mathcal{H}^*, \zeta^* f\}_T$ ,  $f \in I_N$ , which belong to  $\zeta^* C^\infty(V^*Q)$ . If the collection of primary and secondary constraints is not closed with respect to the relation (5.4), let us add the tertiary constraints  $\{\mathcal{H}^*, \{\mathcal{H}^*, \zeta^* f_a\}_T\}_T$  and so on.



Let us assume that  $N$  is a final constraint space for a Hamiltonian form  $H$ . If a Hamiltonian form  $H$  satisfies the relation (5.4), so is a Hamiltonian form

$$H_f = H - f dt \quad (5.5)$$

where  $f \in I'_N$  is a first class constraint. Though Hamiltonian forms  $H$  and  $H_f$  coincide with each other on the constraint space  $N$ , the corresponding Hamilton equations have different solutions on the constraint space  $N$  because  $dH|_N \neq dH_f|_N$ . At the same time,  $d(i_N^* H) = d(i_N^* H_f)$ . Therefore, let us introduce the constrained Hamiltonian form

$$H_N = i_N^* H_f \quad (5.6)$$

which is the same for all  $f \in I'_N$ . Note that  $H_N$  (5.6) is not a true Hamiltonian form on  $N \rightarrow \mathbf{R}$  in general. On sections  $r$  of the fibre bundle  $N \rightarrow \mathbf{R}$ , we can write the equations

$$r^*(u_N]dH_N) = 0, \quad (5.7)$$

where  $u_N$  is an arbitrary vertical vector field on  $N \rightarrow \mathbf{R}$ . They are called the constrained Hamilton equations.

**Proposition 5.1.** *For any Hamiltonian form  $H_f$  (5.5), every solution of the Hamilton equations which lives in the constraint space  $N$  is a solution of the constrained Hamilton equations (5.7).*

*Proof.* The constrained Hamilton equations can be written as

$$r^*(u_N]di_N^* H_f) = r^*(u_N]dH_f|_N) = 0. \quad (5.8)$$

They differ from the Hamilton equations (4.13) for  $H_f$  restricted to  $N$  which read

$$r^*(u]dH_f|_N) = 0, \quad (5.9)$$

where  $r$  is a section of  $N \rightarrow \mathbf{R}$  and  $u$  is an arbitrary vertical vector field on  $V^*Q \rightarrow \mathbf{R}$ . A solution  $r$  of the equations (5.9) satisfies obviously the weaker condition (5.8).  $\square$

*Remark 5.2.* One also can consider the problem of constructing a generalized Hamiltonian system, similar to that for Dirac constraint system in conservative mechanics [23]. Let  $H$  satisfies the condition  $\{\mathcal{H}^*, \zeta^* I'_N\}_T \subset I_N$ , whereas  $\{\mathcal{H}^*, \zeta^* I'_N\}_T \not\subset I_N$ . The goal is to find a constraint  $f \in I_N$  such that the modified Hamiltonian  $H - f dt$  would satisfy both the conditions

$$\{\mathcal{H}^* + \zeta^* f, \zeta^* I'_N\}_T \subset \zeta^* I_N, \quad \{\mathcal{H}^* + \zeta^* f, \zeta^* I_N\}_T \subset \zeta^* I_N.$$

The first of them is fulfilled for any  $f \in I_N$ , while the latter is an equation for a second-class constraint  $f$ .

It should be emphasized that, in contrast with the conservative case, the Hamiltonian vector fields  $\vartheta_f$  for the first class constraints  $f \in I'_N$  in time-dependent mechanics are not generators of gauge symmetries of a Hamiltonian form in general. At the same time, generators of gauge symmetries define an ideal of constraints as follows.

The above construction, except the isomorphism (5.1), can be applied to any ideal  $I$  of  $C^\infty(V^*Q)$ . Then one says that the Poisson algebra  $\overline{I}/I'$  is the reduction of the Poisson algebra  $C^\infty(V^*Q)$  via the ideal  $I$  [18]. In particular, an ideal  $I$  is said to be coisotropic if it is a Poisson algebra. In this case,  $I$  is a Poisson subalgebra of the normalize  $\overline{I}$  (5.2), and coincides with  $I'$  (5.3).

Let  $\mathcal{A}$  be a Lie algebra of generators  $u$  of gauge symmetries of a Hamiltonian form  $H$ . In accordance with the relation (4.23), the corresponding symmetry currents  $J_u$  (4.20) on  $V^*Q$  constitute a Lie algebra with respect to the canonical Poisson bracket on  $V^*Q$ . Let  $I_{\mathcal{A}}$  denotes the ideal of  $C^\infty(V^*Q)$  generated by these symmetry currents. It is readily observed that this ideal is coisotropic. Then one can think of  $I_{\mathcal{A}}$  as being an ideal of first class constraints compatible with the Hamiltonian form  $H$ , i.e.,

$$\{\mathcal{H}^*, \zeta^* I_{\mathcal{A}}\}_T \subset \zeta^* I_{\mathcal{A}}. \quad (5.10)$$

Note that any Hamiltonian form  $H_u = H - J_u dt$ ,  $u \in \mathcal{A}$ , obeys the same relation (5.10), but other currents  $J_{u'}$  are not conserved with respect  $H_u$  if  $[u, u'] \neq 0$ .

Let now  $\mathcal{A}$  be an arbitrary Lie algebra of vertical vector fields  $u$  on the configuration bundle  $Q \rightarrow \mathbf{R}$ . The relation (4.23) remains true, while the corresponding symmetry currents  $J_u$  (4.21) on  $V^*Q$  constitute a Lie algebra and generate the corresponding coisotropic ideal  $I_{\mathcal{A}}$  of  $C^\infty(V^*Q)$  with respect to the canonical Poisson bracket on  $V^*Q$ .

**Proposition 5.2.** *Let  $\mathcal{A}$  be a finite-dimensional Lie algebra of vertical vector fields on the configuration bundle  $Q \rightarrow \mathbf{R}$ . If there exists a reference frame  $\Gamma$  on  $Q \rightarrow \mathbf{R}$  such that  $[\Gamma, \mathcal{A}] = 0$ , then there exists a non-frame Hamiltonian form  $H$  on the Legendre bundle  $V^*Q$  such that  $\mathcal{A}$  is the algebra of gauge symmetries of  $H$ .*

*Proof.* Let  $\overline{\mathcal{A}}$  be the universal enveloping algebra of the Lie algebra of the symmetry currents  $J_u$ ,  $u \in \mathcal{A}$ , (4.21). Then each non-zero element  $C$  of its center of order

$> 1$  can be written as a polynomial in  $J_u$ , and defines the desired Hamiltonian form  $H = H_\Gamma - Cdt$ .  $\square$

## 6 Lagrangian constraints

Let us consider the Hamiltonian description of Lagrangian mechanical systems on a configuration bundle  $Q \rightarrow \mathbf{R}$ . If a Lagrangian is degenerate, we have the Lagrangian constraint subspace of the Legendre bundle  $V^*Q$  and a set of Hamiltonian forms associated with the same Lagrangian. Given a Lagrangian  $L$  (1.6) on the velocity phase space  $J^1Q$ , a Hamiltonian form  $H$  on the momentum phase space  $V^*Q$  is said to be associated with  $L$  if  $H$  satisfies the relations

$$\widehat{L} \circ \widehat{H} \circ \widehat{L} = \widehat{L}, \quad (6.1a)$$

$$H = H_{\widehat{H}} + \widehat{H}^* L \quad (6.1b)$$

where  $\widehat{H}$  and  $\widehat{L}$  are the Hamiltonian morphism (4.10) and the Legendre map (1.17), respectively. A glance at the relation (6.1a) shows that  $\widehat{L} \circ \widehat{H}$  is the projector

$$p_i(z) = \pi_i(t, q^i, \partial^j \mathcal{H}(z)), \quad z \in N_L, \quad (6.2)$$

from  $\Pi$  onto the Lagrangian constraint space  $N_L = \widehat{L}(J^1Y)$ . Accordingly,  $\widehat{H} \circ \widehat{L}$  is the projector from  $J^1Y$  onto  $\widehat{H}(N_L)$ . A Hamiltonian form is called weakly associated with a Lagrangian  $L$  if the condition (6.1b) holds on the Lagrangian constraint space  $N_L$ .

**Proposition 6.1.** [12]. *If a bundle morphism  $\Phi : V^*Q \xrightarrow{Q} J^1Q$  obeys the relation (6.1a), then the Hamiltonian form  $H = -\Phi] \Theta + \Phi^* L$  is weakly associated with the Lagrangian  $L$ . If  $\Phi = \widehat{H}$ , then  $H$  is associated with  $L$ .*

**Lemma 6.2.** *Any Hamiltonian form  $H$  weakly associated with a Lagrangian  $L$  obeys the relation*

$$H|_{N_L} = \widehat{H}^* H_L|_{N_L}, \quad (6.3)$$

where  $H_L$  is the Poincaré–Cartan form (3.2).

*Proof.* The relation (6.1b) takes the coordinate form

$$\mathcal{H}(z) = p_i \partial^i \mathcal{H} - \mathcal{L}(t, q^i, \partial^j \mathcal{H}(z)), \quad z \in N_L. \quad (6.4)$$

Substituting (6.2) and (6.4) in (1.15), we obtain the relation (6.3).  $\square$

The difference between associated and weakly associated Hamiltonian forms lies in the following. Let  $H$  be an associated Hamiltonian form, i.e., the equality (6.4) holds everywhere on  $V^*Q$ . The exterior differential of this equality leads to the relations

$$\begin{aligned}\partial_t \mathcal{H}(z) &= -(\partial_t \mathcal{L}) \circ \widehat{H}(z), & \partial_i \mathcal{H}(z) &= -(\partial_i \mathcal{L}) \circ \widehat{H}(z), & z &\in N_L, \\ (p_i - (\partial_i^t \mathcal{L})(t, q^i, \partial_t^j \mathcal{H})) \partial_t^i \partial_t^a \mathcal{H} &= 0.\end{aligned}$$

The last of them shows that the Hamiltonian form is not regular outside the Lagrangian constraint space  $N_L$ . In particular, any Hamiltonian form is weakly associated with the Lagrangian  $L = 0$ , while the associated Hamiltonian forms are only  $H_\Gamma$ .

Here we restrict our consideration to almost regular Lagrangians  $L$ , i.e., if: (i) the Lagrangian constraint space  $N_L$  is a closed imbedded subbundle  $i_N : N_L \rightarrow V^*Q$  of the bundle  $V^*Q \rightarrow Q$ , (ii) the Legendre map  $\widehat{L} : J^1Q \rightarrow N_L$  is a fibred manifold, and (iii) the pre-image  $\widehat{L}^{-1}(z)$  of any point  $z \in N_L$  is a connected submanifold of  $J^1Q$ .

**Proposition 6.3.** *As an immediate consequence of the above conditions (i), (ii) and Proposition 6.1, a Hamiltonian form  $H$  weakly associated with an almost regular Lagrangian  $L$  exists iff the fibred manifold  $J^1V^*Q \rightarrow N_L$  admits a global section.*

The condition (iii) leads to the following property.

**Lemma 6.4.** [12, 23]. *The Poincaré–Cartan form  $H_L$  for an almost regular Lagrangian  $L$  is constant on the connected pre-image  $\widehat{L}^{-1}(z)$  of any point  $z \in N_L$ .*

An immediate consequence of this fact is the following assertion.

**Proposition 6.5.** [12]. *All Hamiltonian forms weakly associated with an almost regular Lagrangian  $L$  coincide with each other on the Lagrangian constraint space  $N_L$ , and the Poincaré–Cartan form  $H_L$  (3.2) for  $L$  is the pull-back*

$$H_L = \widehat{L}^* H, \quad \pi_i q_t^i - \mathcal{L} = \mathcal{H}(t, q^j, \pi_j), \quad (6.5)$$

*of any such a Hamiltonian form  $H$ .*

It follows that, given Hamiltonian forms  $H$  and  $H'$  weakly associated with an almost regular Lagrangian  $L$ , their difference is  $f dt$ ,  $f \in I_N$ . However,  $\widehat{H}|_{N_L} \neq$

$\widehat{H}'|_{N_L}$  in general. Therefore, the Hamilton equations for  $H$  and  $H'$  do not coincide necessarily on the Lagrangian constraint space  $N_L$ . Their solutions can leave  $N_L$ , i.e., the relation (5.4) fails to hold in general.

Proposition 6.5 enables us to connect Lagrange and Cartan equations for an almost regular Lagrangian  $L$  with the Hamilton equations for Hamiltonian forms weakly associated with  $L$  [12].

**Theorem 6.6.** *Let a section  $r$  of  $V^*Q \rightarrow \mathbf{R}$  be a solution of the Hamilton equations (4.12a) – (4.12b) for a Hamiltonian form  $H$  weakly associated with an almost regular Lagrangian  $L$ . If  $r$  lives in the constraint space  $N_L$ , the section  $c = \pi_Q \circ r$  of  $Q \rightarrow \mathbf{R}$  satisfies the Lagrange equations (3.4), while  $\bar{c} = \widehat{H} \circ r$  obeys the Cartan equations (3.9).*

The proof is based on the relation

$$\mathcal{E}_{\bar{L}} = (J^1 \widehat{L})^* \mathcal{E}_H \quad (6.6)$$

or on the equivalent relation  $\bar{L} = (J^1 \widehat{L})^* L_H$  which are derived from the equality (6.5). The converse assertion is more intricate.

**Theorem 6.7.** *Given an almost regular Lagrangian  $L$ , let a section  $\bar{c}$  of the jet bundle  $J^1 Q \rightarrow \mathbf{R}$  be a solution of the Cartan equations (3.9). Let  $H$  be a Hamiltonian form weakly associated with  $L$ , and let  $H$  satisfy the relation*

$$\widehat{H} \circ \widehat{L} \circ \bar{c} = J^1(\pi_0^1 \circ \bar{c}). \quad (6.7)$$

*Then, the section  $r = \widehat{L} \circ \bar{c}$  of the Legendre bundle  $V^*Q \rightarrow \mathbf{R}$  is a solution of the Hamilton equations (4.12a) – (4.12b) for  $H$ .*

*Remark 6.1.* Since  $\widehat{H} \circ \widehat{L}$  in Theorem (6.7) is a projection operator, the condition (6.7) implies that the solution  $\bar{s}$  of the Cartan equations is actually an integrable section  $\bar{c} = \dot{c}$  where  $c$  is a solution of the Lagrange equations. In fact, the relation (6.6) gives more than it is needed for proving Theorem 6.6. Using this relation, one can justify that, if  $\gamma$  is a Hamiltonian connection for a Hamiltonian form  $H$  weakly associated with an almost regular Lagrangian  $L$ , then the composition  $J^1 \widehat{H} \circ \gamma \circ \widehat{L}$  takes its values in  $\text{Ker } \mathcal{E}_{\bar{L}} \cap J^2 Y$ , i.e., this is a local holonomic Lagrangian connection on  $\widehat{H}(N_L)$  [12]. A converse of this assertion, however, fails to be true in the case of degenerate Lagrangians. Let a Lagrangian  $L$  be hyperregular, i.e., the Legendre map  $\widehat{L}$  is a diffeomorphism. Then  $\widehat{L}^{-1}$  is a Hamiltonian map, and there is a unique

Hamiltonian form  $H = H_{\widehat{L}^{-1}} + \widehat{L}^{-1*}L$  weakly associated with  $L$ . In this case, both the relation (6.6) and the converse one  $\mathcal{E}_H = (J^1\widehat{H})^*\mathcal{E}_{\widehat{L}}$  hold. It follows that the Lagrange equations for  $L$  and the Hamilton equations for  $H$  are equivalent.

We will say that a set of Hamiltonian forms  $H$  weakly associated with an almost regular Lagrangian  $L$  is complete if, for each solution  $c$  of the Lagrange equations, there exists a solution  $r$  of the Hamilton equations for a Hamiltonian form  $H$  from this set such that  $c = \pi_Q \circ r$ . By virtue of Theorem 6.7 and Remark 6.1, a set of weakly associated Hamiltonian forms is complete if, for every solution  $c$  on  $\mathbf{R}$  of the Lagrange equations for  $L$ , there is a Hamiltonian form  $H$  from this set which fulfills the relation

$$\widehat{H} \circ \widehat{L} \circ \dot{c} = \dot{c}. \quad (6.8)$$

In accordance with Proposition 6.3, on an open neighbourhood in  $V^*Q$  of each point  $z \in N_L$ , there exists a complete set of local Hamiltonian forms weakly associated with an almost regular Lagrangian  $L$ . Moreover, one can always construct a complete set of associated local Hamiltonian forms [29, 35]

Given a Hamiltonian form  $H$  weakly associated with an almost regular Lagrangian  $L$ , let us consider the corresponding constrained Hamiltonian form  $H_N$  (5.6). By virtue of Proposition (6.5),  $H_N$  is the same for all Hamiltonian forms weakly associated with  $L$ , and  $H_L = \widehat{L}^*H_N$ . The first of these facts leads to the assertion proved similarly to Proposition 5.1.

**Proposition 6.8.** *For any Hamiltonian form  $H$  weakly associated with an almost regular Lagrangian  $L$ , every solution of the Hamilton equations which lives in the Lagrangian constraint space  $N_L$  is a solution of the constrained Hamilton equations (5.7).*

Using the equality  $H_L = \widehat{L}^*H_N$ , one can show that the constrained Hamilton equations (5.7) are equivalent to the Hamilton–De Donder equations (3.12) and, by virtue of Theorem 3.1, are quasi-equivalent to the Cartan equations (3.10) [12, 23].

## 7 Quadratic degenerate systems

Let us study the important case of almost regular quadratic Lagrangians. We show that, in this case, there always exist both a complete set of associated Hamiltonian forms and a complete set of non-degenerate weakly associated Hamiltonian forms. The latter is important for quantization.

Given a configuration bundle  $Q \rightarrow \mathbf{R}$ , let us consider a quadratic Lagrangian  $L$  which has the coordinate expression

$$\mathcal{L} = \frac{1}{2}a_{ij}q_t^i q_t^j + b_i q_t^i + c, \quad (7.1)$$

where  $a$ ,  $b$  and  $c$  are local functions on  $Q$ . This property is coordinate-independent due to the affine transformation law of the coordinates  $q_t^i$ . The associated Legendre map

$$p_i \circ \hat{L} = a_{ij}q_t^j + b_i \quad (7.2)$$

is an affine morphism over  $Q$ . It defines the corresponding linear morphism

$$\bar{L} : VQ \rightarrow_{\bar{Q}} V^*Q, \quad p_i \circ \bar{L} = a_{ij}\dot{q}^j. \quad (7.3)$$

Let the Lagrangian  $L$  (7.1) be almost regular, i.e., the matrix function  $a_{ij}$  is of constant rank. Then the Lagrangian constraint space  $N_L$  (7.2) is an affine subbundle of the bundle  $V^*Q \rightarrow Q$ , modelled over the vector subbundle  $\bar{N}_L$  (7.3) of  $V^*Q \rightarrow Q$ . Hence,  $N_L \rightarrow Q$  has a global section. For the sake of simplicity, let us assume that it is the canonical zero section  $\hat{0}(Q)$  of  $V^*Q \rightarrow Q$ . Then  $\bar{N}_L = N_L$ . Accordingly, the kernel of the Legendre map (7.2) is an affine subbundle of the affine jet bundle  $J^1Q \rightarrow Q$ , modelled over the kernel of the linear morphism  $\bar{L}$  (7.3). Then there exists a connection

$$\Gamma : Q \rightarrow \text{Ker } \hat{L} \subset J^1Q, \quad (7.4)$$

$$a_{ij}\Gamma_\mu^j + b_i = 0, \quad (7.5)$$

on  $Q \rightarrow \mathbf{R}$ . Connections (7.4) constitute an affine space modelled over the linear space of vertical vector fields  $v$  on  $Q \rightarrow \mathbf{R}$ , satisfying the conditions

$$a_{ij}v^j = 0 \quad (7.6)$$

and, as a consequence, the conditions  $v^i b_i = 0$ . If the Lagrangian (7.1) is regular, the connection (7.4) is unique.

The matrix  $a$  in the Lagrangian  $L$  (7.1) can be seen as a degenerate fibre metric of constant rank in  $VQ \rightarrow Q$ . Then it satisfies the following Lemma.

**Lemma 7.1.** *Given a  $k$ -dimensional vector bundle  $E \rightarrow Z$ , let  $a$  be a section of rank  $r$  of the tensor bundle  $\overset{2}{\vee} E^* \rightarrow Z$ . There is a splitting*

$$E = \text{Ker } a \oplus_{\overset{2}{\vee} E^*} E' \quad (7.7)$$

where  $E' = E/\text{Ker } a$  is the quotient bundle, and  $a$  is a non-degenerate fibre metric in  $E'$ .

*Proof.* Since  $a$  exists, the structure group  $GL(k, \mathbf{R})$  of the vector bundle  $E \rightarrow Z$  is reducible to the subgroup  $GL(r, k-r; \mathbf{R})$  of general linear transformations of  $\mathbf{R}^k$  which keep its  $r$ -dimensional subspace, and to its subgroup  $GL(r, \mathbf{R}) \times GL(k-r, \mathbf{R})$ .  
□

**Theorem 7.2.** *There exists a linear bundle map*

$$\sigma : V^*Q \xrightarrow{Q} VQ, \quad \dot{q}^i \circ \sigma = \sigma^{ij} p_j, \quad (7.8)$$

such that  $\bar{L} \circ \sigma \circ i_N = i_N$ .

*Proof.* The map (7.8) is a solution of the algebraic equations

$$a_{ij} \sigma^{jk} a_{kb} = a_{ib}. \quad (7.9)$$

By virtue of Lemma 7.1, there exist the bundle slitting

$$VQ = \text{Ker } a \oplus_Q E' \quad (7.10)$$

and a (non-holonomic) atlas of this bundle such that transition functions of  $\text{Ker } a$  and  $E'$  are independent. Since  $a$  is a non-degenerate fibre metric in  $E'$ , there exists an atlas of  $E'$  such that  $a$  is brought into a diagonal matrix with non-vanishing components  $a_{AA}$ . Due to the splitting (7.10), we have the corresponding bundle splitting

$$V^*Q = (\text{Ker } a)^* \oplus_Q \text{Im } a. \quad (7.11)$$

Then the desired map  $\sigma$  is represented by a direct sum  $\sigma_1 \oplus \sigma_0$  of an arbitrary section  $\sigma_1$  of the bundle  $\overset{2}{V} \text{Ker } a^* \rightarrow Q$  and the section  $\sigma_0$  of the bundle  $\overset{2}{V} E' \rightarrow Q$ , which has non-vanishing components  $\sigma^{AA} = (a_{AA})^{-1}$  with respect to the above mentioned atlas of  $E'$ . Moreover,  $\sigma$  satisfies the particular relations

$$\sigma_0 = \sigma_0 \circ \bar{L} \circ \sigma_0, \quad a \circ \sigma_1 = 0, \quad \sigma_1 \circ a = 0. \quad (7.12)$$

□



**Corollary 7.3.** *The splitting (7.10) leads to the splitting*

$$J^1Q = \mathcal{S}(J^1Q) \oplus_Q \mathcal{F}(J^1Q) = \text{Ker } \widehat{L} \oplus_Q \text{Im}(\sigma \circ \widehat{L}), \quad (7.13a)$$

$$q_t^i = \mathcal{S}^i + \mathcal{F}^i = [q_t^i - \sigma_0^{ik}(a_{kj}q_t^j + b_k)] + [\sigma_0^{ik}(a_{kj}q_t^j + b_k)], \quad (7.13b)$$

while the splitting (7.11) can be written as

$$V^*Q = \mathcal{R}(V^*Q) \oplus_Q \mathcal{P}(V^*Q) = \text{Ker } \sigma_0 \oplus_Q N_L, \quad (7.14a)$$

$$p_i = \mathcal{R}_i + \mathcal{P}_i = [p_i - a_{ij}\sigma_0^{jk}p_k] + [a_{ij}\sigma_0^{jk}p_k]. \quad (7.14b)$$

It is readily observed that, with respect to the coordinates  $\mathcal{S}_\lambda^i$  and  $\mathcal{F}_\lambda^i$  (7.13b), the Lagrangian (7.1) reads

$$\mathcal{L} = \frac{1}{2}a_{ij}\mathcal{F}^i\mathcal{F}^j + c', \quad (7.15)$$

while the Lagrangian constraint space is given by the reducible constraints

$$\mathcal{R}_i = p_i - a_{ij}\sigma_0^{jk}p_k = 0. \quad (7.16)$$

Given the linear map  $\sigma$  (7.8) and the connection  $\Gamma$  (7.4), let us consider the affine Hamiltonian map

$$\Phi = \widehat{\Gamma} + \sigma : V^*Q \rightarrow J^1Q, \quad \Phi^i = \Gamma^i + \sigma^{ij}p_j, \quad (7.17)$$

and the Hamiltonian form

$$\begin{aligned} H = H_\Phi + \Phi^*L &= p_i dq^i - [p_i \Gamma^i + \frac{1}{2}\sigma_0^{ij}p_i p_j + \sigma_1^{ij}p_i p_j - c']dt = \\ &= (\mathcal{R}_i + \mathcal{P}_i) dq^i - [(\mathcal{R}_i + \mathcal{P}_i) \Gamma^i + \frac{1}{2}\sigma_0^{ij}\mathcal{P}_i \mathcal{P}_j + \sigma_1^{ij}p_i p_j - c']dt. \end{aligned} \quad (7.18)$$

In particular, if  $\sigma_1$  is non-degenerate, so is the Hamiltonian form (7.18).

**Theorem 7.4.** *The Hamiltonian forms (7.18) parameterized by connections  $\Gamma$  (7.4) are weakly associated with the Lagrangian (7.1) and constitute a complete set.*

*Proof.* By the very definitions of  $\Gamma$  and  $\sigma$ , the Hamiltonian map (7.17) satisfies the condition (6.1a). Then  $H$  is weakly associated with  $L$  (7.1) in accordance with

Proposition 6.1. Let us write the corresponding Hamilton equations (4.12a) for a section  $r$  of the Legendre bundle  $V^*Q \rightarrow \mathbf{R}$ . They are

$$\dot{c} = (\widehat{\Gamma} + \sigma) \circ r, \quad c = \pi_Q \circ r. \quad (7.19)$$

Due to the surjections  $\mathcal{S}$  and  $\mathcal{F}$  (7.13a), the Hamilton equations (7.19) break in two parts

$$\mathcal{S} \circ \dot{c} = \Gamma \circ c, \quad \dot{r}^i - \sigma^{ik}(a_{kj}\dot{r}^j + b_k) = \Gamma^i \circ c, \quad (7.20)$$

$$\mathcal{F} \circ \dot{c} = \sigma \circ r, \quad \sigma^{ik}(a_{kj}\dot{r}^j + b_k) = \sigma^{ik}r_k. \quad (7.21)$$

Let  $c$  be an arbitrary section of  $Q \rightarrow \mathbf{R}$ , e.g., a solution of the Lagrange equations. There exists a connection  $\Gamma$  (7.4) such that the relation (7.20) holds, namely,  $\Gamma = \mathcal{S} \circ \Gamma'$  where  $\Gamma'$  is a connection on  $Q \rightarrow \mathbf{R}$  which has  $c$  as an integral section. It is easily seen that, in this case, the Hamiltonian map (7.17) satisfies the relation (6.8) for  $c$ . Hence, the Hamiltonian forms (7.18) constitute a complete set.  $\square$

It is readily observed that, if  $\sigma_1 = 0$ , then  $\Phi = \widehat{H}$  and the Hamiltonian forms (7.18) are associated with the Lagrangian (7.1) in accordance with Proposition 6.1. Thus, for different  $\sigma_1$ , we have different complete sets of Hamiltonian forms (7.18). Hamiltonian forms  $H$  (7.18) of such a complete set differ from each other in the term  $v^i \mathcal{R}_i$ , where  $v$  are vertical vector fields (7.6). It follows from the splitting (7.14a) that this term vanishes on the Lagrangian constraint space. The corresponding constrained Hamiltonian form  $H_N = i_N^* H$  and the constrained Hamilton equations (5.7) can be written. In the case of quadratic Lagrangians, we can improve Proposition 6.8 as follows.

**Proposition 7.5.** *For every Hamiltonian form  $H$  (7.18), the Hamilton equations (4.12b) and (7.21) restricted to the Lagrangian constraint space  $N_L$  are equivalent to the constrained Hamilton equations.*

*Proof.* Due to the splitting (7.14a), we have the corresponding splitting of the vertical tangent bundle  $V_Q V^*Q$  of the bundle  $V^*Q \rightarrow Q$ . In particular, any vertical vector field  $u$  on  $V^*Q \rightarrow \mathbf{R}$  admits the decomposition

$$u = [u - u_{TN}] + u_{TN}, \quad u_{TN} = u^i \partial_i + a_{ij} \sigma_0^{jk} u_k \partial^j,$$

such that  $u_N = u_{TN}|_{N_L}$  is a vertical vector field on the Lagrangian constraint space  $N_L \rightarrow \mathbf{R}$ . Let us consider the equations

$$r^*(u_{TN}|dH) = 0 \quad (7.22)$$

where  $r$  is a section of  $V^*Q \rightarrow \mathbf{R}$  and  $u$  is an arbitrary vertical vector field on  $V^*Q \rightarrow \mathbf{R}$ . They are equivalent to the pair of equations

$$r^*(a_{ij}\sigma_0^{jk}\partial^i]dH) = 0, \quad (7.23a)$$

$$r^*(\partial_i]dH) = 0. \quad (7.23b)$$

The equations (7.23b) are obviously the Hamilton equations (4.12b) for  $H$ . Bearing in mind the relations (7.5) and (7.12), one can easily show that the equations (7.23a) coincide with the Hamilton equations (7.21). The proof is completed by observing that, restricted to the Lagrangian constraint space  $N_L$ , the equations (7.22) are exactly the constrained Hamilton equations (5.8).  $\square$

Proposition 7.5 shows that, restricted to the Lagrangian constraint space, the Hamilton equations for different Hamiltonian forms (7.18) associated with the same quadratic Lagrangian (7.1) differ from each other in the equations (7.20). These equations are independent of momenta and play the role of gauge-type conditions.

We aim to obtain the Koszul–Tate resolution for the constraints (7.16). Since these constraints are not necessarily irreducible, we need an infinite number of ghosts and antighosts [10, 18].

## 8 Simple BRST manifolds

Let  $E = E_0 \oplus E_1 \rightarrow Z$  be the Whitney sum of vector bundles  $E_0 \rightarrow Z$  and  $E_1 \rightarrow Z$  over a paracompact manifold  $Z$ . One can think of  $E$  as being a bundle of vector superspaces with a typical fibre  $V = V_0 \oplus V_1$  where transition functions of  $E_0$  and  $E_1$  are independent. Let us consider the exterior bundle

$$\wedge E^* = \bigoplus_{k=0}^{\infty} (\wedge_Z^k E^*), \quad (8.1)$$

which is the tensor bundle  $\otimes E^*$  modulo elements

$$e_0 e'_0 - e'_0 e_0, \quad e_1 e'_1 + e'_1 e_1, \quad e_0 e_1 - e_1 e_0 \quad e_0, e'_0 \in E_{0z}^*, \quad e_1, e'_1 \in E_{1z}^*, \quad z \in Z.$$

$\wedge E^*$  is the bundle of commutative superalgebras  $\wedge V$  which is the tensor product  $\vee E_0^* \otimes \wedge E_1^*$  modulo elements

$$e_0 e_1 - e_1 e_0 \quad e_0 \in E_{0z}^*, \quad e_1 \in E_{1z}^*, \quad z \in Z.$$

The global sections of  $\wedge E^*$  constitute a commutative superalgebra  $\mathcal{A}(Z)$  over the free  $C^\infty(Z)$ -module  $E_0^*(Z) \oplus E_1^*(Z)$  of global sections of  $E^*$ . This is the product of the commutative algebra  $\mathcal{A}_0(Z)$  of global sections of  $\vee E_0^* \rightarrow Z$  and the graded algebra  $\mathcal{A}_1(Z)$  of global sections of the Grassman bundle  $\wedge E_1^* \rightarrow Z$ . We use the notation  $[.]$  for the Grassman parity.

*Remark 8.1.* Let  $\mathcal{A}_1$  be the sheaf of sections of the Grassman bundle  $\wedge E_1^*$ . The pair  $(Z, \mathcal{A}_1)$  is a graded manifold [1]. By the well-known Batchelor theorem, every graded manifold is isomorphic to a sheaf of sections of some Grassman bundle, but not in a canonical way. Therefore, the construction below can be extended to an arbitrary commutative superalgebra over a free  $C^\infty(Z)$ -module  $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$  of finite rank. We call  $(Z, \mathcal{A})$  a BRST manifold, while sections of  $\wedge E^*$  are said to be BRST functions.

Let us study the  $\mathcal{A}(Z)$ -module  $\text{Der}\mathcal{A}(Z)$  of graded derivations of  $\mathcal{A}(Z)$ . Recall that by a graded derivation of the commutative superalgebra  $\mathcal{A}(Z)$  is meant an endomorphism of  $\mathcal{A}(Z)$  such that

$$u(ff') = u(f)f' + (-1)^{[u][f]}fu(f') \quad (8.2)$$

for the homogeneous elements  $u \in \text{Der}\mathcal{A}(Z)$  and  $f, f' \in \mathcal{A}(Z)$ .

**Proposition 8.1.** *Graded derivations (8.2) are represented by sections of a vector bundle.*

*Proof.* Let  $\{c^a\}$  be the holonomic bases for  $E^* \rightarrow Z$  with respect to some bundle atlas  $(z^A, v^i)$  of  $E \rightarrow Z$  with transition functions  $\{\rho_b^a\}$ , i.e.,  $c'^a = \rho_b^a(z)c^b$ . Then BRST functions read

$$f = \sum_{k=0} \frac{1}{k!} f_{a_1 \dots a_k} c^{a_1} \dots c^{a_k}, \quad (8.3)$$

where  $f_{a_1 \dots a_k}$  are local functions on  $Z$ , and we omit the symbol of an exterior product of elements  $c$ . The coordinate transformation law of BRST functions (8.3) is obvious. Due to the canonical splitting  $VE = E \times E$ , the vertical tangent bundle  $VE \rightarrow E$  can be provided with the fibre bases  $\{\partial_a\}$  dual of  $\{c^a\}$ . These are fibre bases for  $\text{pr}_2 VE = E$ . Then any derivation  $u$  of  $\mathcal{A}(U)$  on a trivialization domain  $U$  of  $E$  reads

$$u = u^A \partial_A + u^a \partial_a, \quad (8.4)$$

where  $u^A, u^a$  are local BRST functions and  $u$  acts on  $f \in \mathcal{A}(U)$  by the rule

$$u(f_{a \dots b} c^a \dots c^b) = u^A \partial_A (f_{a \dots b}) c^a \dots c^b + u^a f_{a \dots b} \partial_a (c^a \dots c^b). \quad (8.5)$$

This rule implies the corresponding coordinate transformation law

$$u'^A = u^A, \quad u'^a = \rho_j^a u^j + u^A \partial_A (\rho_j^a) c^j \quad (8.6)$$

of derivations (8.4). Let us consider the vector bundle  $\mathcal{V}_E \rightarrow Z$  which is locally isomorphic to the vector bundle

$$\mathcal{V}_E|_U \approx \wedge E^* \otimes_Z (\text{pr}_2 V E \oplus_Z T Z)|_U,$$

and has the transition functions

$$\begin{aligned} z'^A_{i_1 \dots i_k} &= \rho^{-1}_{i_1}{}^{a_1} \dots \rho^{-1}_{i_k}{}^{a_k} z^A_{a_1 \dots a_k}, \\ v'^i_{j_1 \dots j_k} &= \rho^{-1}_{j_1}{}^{b_1} \dots \rho^{-1}_{j_k}{}^{b_k} \left[ \rho^i_j v^j_{b_1 \dots b_k} + \frac{k!}{(k-1)!} z^A_{b_1 \dots b_{k-1}} \partial_A (\rho^i_{b_k}) \right] \end{aligned}$$

of the bundle coordinates  $(z^A_{a_1 \dots a_k}, v^i_{b_1 \dots b_k})$ ,  $k = 0, \dots$ . These transition functions fulfill the cocycle relations. It is readily observed that, for any trivialization domain  $U$ , the  $\mathcal{A}$ -module  $\text{Der} \mathcal{A}(U)$  with the transition functions (8.6) is isomorphic to the  $\mathcal{A}$ -module of local sections of  $\mathcal{V}_E|_U \rightarrow U$ . One can show that, if  $U' \subset U$  are open sets, there is the restriction morphism  $\text{Der} \mathcal{A}(U) \rightarrow \text{Der} \mathcal{A}(U')$ . It follows that, restricted to an open subset  $U$ , every derivation  $u$  of  $\mathcal{A}(Z)$  coincides with some local section  $u_U$  of  $\mathcal{V}_E|_U \rightarrow U$ , whose collection  $\{u_U, U \subset Z\}$  defines uniquely a global section of  $\mathcal{V}_E \rightarrow Z$ , called a BRST vector field on  $Z$ . BRST vector fields constitute a Lie superalgebra with respect to the bracket

$$[u, u'] = uu' + (-1)^{[u][u'] + 1} u' u.$$

□

**Corollary 8.2.** *The sheaf of sections of  $\mathcal{V}_E \rightarrow Z$  is isomorphic to the sheaf of graded derivations of the sheaf  $\mathcal{A}$ .*

There is the exact sequence over  $Z$  of vector bundles

$$0 \rightarrow \wedge E^* \otimes_Z \text{pr}_2 V E \rightarrow \mathcal{V}_E \rightarrow \wedge E^* \otimes_Z T Z \rightarrow 0. \quad (8.7)$$

Its splitting

$$\tilde{\gamma} : \dot{z}^A \partial_A \mapsto \dot{z}^A (\partial_A + \tilde{\gamma}_A^a \partial_a) \quad (8.8)$$

transforms every vector field  $\tau$  on  $Z$  into a BRST vector field

$$\tau = \tau^A \partial_A \mapsto \nabla_\tau = \tau^A (\partial_A + \tilde{\gamma}_A^a \partial_a),$$

which is the derivation  $\nabla_\tau$  of  $\mathcal{A}(Z)$  such that

$$\nabla_\tau(sf) = (\tau \rfloor ds)f + s\nabla_\tau(f), \quad f \in \mathcal{A}(Z), \quad s \in C^\infty(Z).$$

Thus, one can think of the splitting (8.8) as being a BRST connection on  $Z$ . For instance, every linear connection

$$\gamma = dz^A \otimes (\partial_A + \gamma_A^a v^b \partial_a)$$

on the vector bundle  $E \rightarrow Z$  yields the BRST connection

$$\gamma_S = dz^A \otimes (\partial_A + \gamma_A^a c^b \partial_a) \tag{8.9}$$

on  $Z$  such that, for any vector field  $\tau$  on  $Z$  and any BRST function  $f$ , the graded derivation  $\nabla_\tau(f)$  is exactly the covariant derivative of  $f$  relative to the connection  $\gamma$ .

The  $\wedge E^*$ -dual  $\mathcal{V}_E^*$  of  $\mathcal{V}_E$  is a vector bundle over  $Z$  which is locally isomorphic to the vector bundle

$$\mathcal{V}_E^*|_U \approx \wedge E^* \otimes_Z (\text{pr}_2 V E^* \oplus_Z T^* Z)|_U,$$

and has the transition functions

$$\begin{aligned} v'_{j_1 \dots j_k j} &= \rho_{j_1}^{-1a_1} \cdots \rho_{j_k}^{-1a_k} \rho_j^{-1a} v_{a_1 \dots a_k a}, \\ z'_{i_1 \dots i_k A} &= \rho_{i_1}^{-1b_1} \cdots \rho_{i_k}^{-1b_k} \left[ z_{b_1 \dots b_k A} + \frac{k!}{(k-1)!} v_{b_1 \dots b_k j} \partial_A(\rho_j^j) \right] \end{aligned}$$

of the bundle coordinates  $(z_{a_1 \dots a_k A}, v_{b_1 \dots b_k j})$ ,  $k = 0, \dots$ , with respect to the dual bases  $\{dz^A\}$  for  $T^*Z$  and  $\{dc^b\}$  for  $\text{pr}_2 V^* E = E^*$ . Global sections of this vector bundle constitute the  $\mathcal{A}(Z)$ -module of exterior BRST 1-forms  $\phi = \phi_A dz^A + \phi_a dc^a$  on  $Z$ , which have the coordinate transformation law

$$\phi'_a = \rho^{-1b}_a \phi_b, \quad \phi'_A = \phi_A + \rho^{-1b}_a \partial_A(\rho_j^a) \phi_b c^j.$$

Then the morphism  $\phi : u \rightarrow \mathcal{A}(Z)$  can be seen as the interior product

$$u \rfloor \phi = u^A \phi_A + (-1)^{[\phi_a]} u^a \phi_a. \tag{8.10}$$

There is the exact sequence

$$0 \rightarrow \wedge E^* \otimes_Z T^* Z \rightarrow \mathcal{V}_E^* \rightarrow \wedge E^* \otimes_Z \text{pr}_2 V E^* \rightarrow 0. \quad (8.11)$$

Any BRST connection  $\tilde{\gamma}$  (8.8) yields the splitting of the exact sequence (8.11), and defines the corresponding decomposition of BRST 1-forms

$$\phi = \phi_A dz^A + \phi_a dc^a = (\phi_A + \phi_a \tilde{\gamma}_A^a) dz^A + \phi_a (dc^a - \tilde{\gamma}_A^a dz^A).$$

BRST  $k$ -forms  $\phi$  can be defined as sections of the graded exterior bundle  $\overline{\wedge}_Z^k \mathcal{V}_E^*$  such that

$$\phi \overline{\wedge} \sigma = (-1)^{|\phi||\sigma| + [\phi][\sigma]} \sigma \overline{\wedge} \phi.$$

The interior product (8.10) is extended to higher BRST forms by the rule

$$u \rfloor (\phi \overline{\wedge} \sigma) = (u \rfloor \phi) \overline{\wedge} \sigma + (-1)^{|\phi| + [\phi][u]} \phi \overline{\wedge} (u \rfloor \sigma).$$

The graded exterior differential  $d$  of BRST functions is introduced by the condition  $u \rfloor df = u(f)$  for an arbitrary BRST vector field  $u$ , and is extended uniquely to higher BRST forms by the rules

$$d(\phi \overline{\wedge} \sigma) = (d\phi) \overline{\wedge} \sigma + (-1)^{|\phi|} \phi \overline{\wedge} (d\sigma), \quad d \circ d = 0.$$

It takes the coordinate form

$$d\phi = dz^A \overline{\wedge} \partial_A(\phi) + dc^a \overline{\wedge} \partial_a(\phi),$$

where the left derivatives  $\partial_A, \partial_a$  act on the coefficients of BRST forms by the rule (8.5), and they are graded commutative with the forms  $dz^A, dc^a$ . The Lie derivative of a BRST form  $\phi$  along a BRST vector field  $u$  is given by the familiar formula

$$\mathbf{L}_u \phi = u \rfloor d\phi + d(u \rfloor \phi).$$

## 9 The Koszul–Tate resolution

To construct the vector bundle  $E$  of antighosts, let us consider the vertical tangent bundle  $V_Q(V^*Q)$  of  $V^*Q \rightarrow Q$ . Let us chose the bundle  $E$  as the Whitney sum of

the bundles  $E_0 \oplus E_1$  over  $V^*Q$  which are the infinite Whitney sum over  $V^*Q$  of the copies of  $V_Q(V^*Q)$ . We have

$$E = V_Q(V^*Q) \bigoplus_{V^*Q} V_Q(V^*Q) \oplus \dots \quad (9.1)$$

This bundle is provided with the holonomic coordinates  $(t, q^i, p_i, \dot{p}_i^{(k)})$ ,  $k = 0, 1, \dots$ , where  $(t, q^i, p_i, \dot{p}_i^{(2r)})$  are coordinates on  $E_0$ , while  $(t, q^i, p_i, \dot{p}_i^{(2r+1)})$  are those on  $E_1$ . We call  $k$  the antighost number, while  $k \bmod 2$  is the Grassman parity. The dual of  $E \rightarrow V^*Q$  is

$$E^* = V_Q^*(V^*Q) \bigoplus_{V^*Q} V_Q^*(V^*Q) \oplus \dots$$

It is endowed with the associated fibre bases  $\{c_i^{(k)}\}$ ,  $k = 1, 2, \dots$ , such that  $c_i^{(k)}$  have the same linear coordinate transformation law as the coordinates  $p_i$ . The corresponding BRST vector fields and BRST forms are introduced on  $V^*Q$  as sections of the vector bundles  $\mathcal{V}_E$  and  $\mathcal{V}_E^*$ , respectively.

The  $C^\infty(V^*Q)$ -module  $\mathcal{A}(V^*Q)$  of BRST functions is graded by the antighost number as

$$\mathcal{A}(V^*Q) = \bigoplus_{k=0}^{\infty} \mathcal{N}^k, \quad \mathcal{N}^0 = C^\infty(V^*Q).$$

Its terms  $\mathcal{N}^k$  constitute a complex

$$0 \longleftarrow C^\infty(V^*Q) \longleftarrow \mathcal{N}^1 \longleftarrow \dots \quad (9.2)$$

with respect to the Koszul–Tate differential

$$\begin{aligned} \delta : C^\infty(V^*Q) &\rightarrow 0, \\ \delta(c_i^{2r}) &= a_{ij}\sigma^{jk}c_k^{2r-1}, \quad r > 0, \\ \delta(c_i^{2r+1}) &= (\delta_i^k - a_{ij}\sigma^{jk})c_k^{2r}, \quad r > 0, \\ \delta(c_i^1) &= (\delta_i^k - a_{ij}\sigma^{jk})p_k. \end{aligned} \quad (9.3)$$

The nilpotency property  $\delta \circ \delta = 0$  of this differential is the corollary of the relations (7.9) and (7.12).

**Proposition 9.1.** *The complex (9.2) with respect to the differential (9.3) is the Koszul–Tate resolution, i.e., its homology groups are*

$$H_{k>1} = 0, \quad H_0 = C^\infty(V^*Q)/I_N = C^\infty(N_L).$$



Note that, in particular cases of the degenerate quadratic Lagrangian (7.1), the complex (9.2) may have a subcomplex, which is also the Koszul–Tate resolution. For instance, if the fibre metric  $a$  in  $VQ \rightarrow Q$  is diagonal with respect to a holonomic atlas of  $VQ$ , the constraints (7.16) are irreducible and the complex (9.2) contains a subcomplex which consists only of the antighosts  $c_i^{(1)}$ .

Now let us construct the BRST charge  $\mathbf{Q}$  such that

$$\delta(f) = \{\mathbf{Q}, f\}, \quad f \in \mathcal{A}(V^*Q)$$

with respect to some Poisson bracket. The problem is to find the Poisson bracket such that  $\{f, g\} = 0$  for all  $f, g \in C^\infty(V^*Q)$ . To overcome this difficulty, one can consider the vertical extension of Hamiltonian formalism onto the configuration bundle  $VQ \rightarrow \mathbf{R}$  [13, 23].

**Lemma 9.2.** *Given a fibre bundle  $Y \rightarrow X$ , there is the isomorphism*

$$VV^*Y \underset{VY}{\cong} V^*VY, \quad p_i \longleftrightarrow \dot{v}_i, \quad \dot{p}_i \longleftrightarrow \dot{y}_i.$$

*Proof.* The proof is based on inspection of the transformation laws of the holonomic coordinates  $(x^\lambda, y^i, p_i)$  on  $V^*Y$  and  $(x^\lambda, y^i, v^i)$  on  $VY$ .  $\square$

Given a configuration bundle  $Q \rightarrow \mathbf{R}$ , let us consider the vertical tangent bundle  $VQ \rightarrow \mathbf{R}$ , seen as a configuration bundle of the above mentioned vertical extension of Hamiltonian formalism. By virtue of Lemma 9.2, the corresponding Legendre bundle  $V^*(VQ)$  is isomorphic to  $V(V^*Q)$ , and is provided with the holonomic coordinates  $(t, q^i, p_i, \dot{q}^i, \dot{p}_i)$  such that  $(q^i, \dot{p}_i)$  and  $(\dot{q}^i, p_i)$  are conjugate pairs of canonical coordinates. The momentum phase space  $V(V^*Q)$  is endowed with the canonical exterior 3-form

$$\Omega_V = \partial_V \Omega = [d\dot{p}_i \wedge dq^i + dp_i \wedge d\dot{q}^i] \wedge dt, \quad (9.4)$$

where we use the compact notation

$$\dot{\partial}_i = \frac{\partial}{\partial \dot{q}^i}, \quad \partial^i = \frac{\partial}{\partial \dot{p}_i}, \quad \partial_V = \dot{q}^i \partial_i + \dot{p}_i \partial^i.$$

The corresponding Poisson bracket on  $V(V^*Q)$  reads

$$\{f, g\}_{VV} = \dot{\partial}^i f \partial_i g + \partial^i f \dot{\partial}_i g - \partial^i g \dot{\partial}_i f - \dot{\partial}^i g \partial_i f.$$

To extend this bracket to BRST functions, let us consider the following graded extension of Hamiltonian formalism [16, 23]. We will assume that  $Q \rightarrow \mathbf{R}$  is a vector bundle, and will further denote  $\Pi = V^*Q$ .

Let us consider the vertical tangent bundle  $V\Pi$ . It admits the canonical decomposition

$$V\Pi = V\Pi \oplus_{\mathbf{R}} V\Pi \xrightarrow{\text{pr}_1} V\Pi. \quad (9.5)$$

Let choose the bundle  $E$  as the Whitney sum of the bundles  $E_0 \oplus E_1$  over  $V\Pi$  which are the infinite Whitney sum over  $V\Pi$  of the copies of  $V\Pi$ . In view of the decomposition (9.5), we have

$$E = V\Pi \oplus_{\mathbf{R}} V\Pi \oplus \dots \xrightarrow{\text{pr}_1} V\Pi.$$

This bundle is provided with the holonomic coordinates  $(t, q^i, p_i, \dot{q}_{(k)}^i, \dot{p}_i^{(k)})$ ,  $k = 0, 1, \dots$ , where  $(t, q^i, p_i, \dot{q}_{(2r)}^i, \dot{p}_i^{(2r)})$  are coordinates on  $E_0$  and  $(t, q^i, p_i, \dot{q}_{(2r+1)}^i, \dot{p}_i^{(2r+1)})$  are those on  $E_1$ . The dual of  $E \rightarrow V\Pi$  is

$$E^* = V\Pi \oplus_{\mathbf{R}} V\Pi^* \oplus \dots.$$

It is endowed with the associated fibre bases  $\{\bar{c}_{(k)}^i, \bar{c}_i^{(k)}, c_{(k)}^i, c_i^{(k)}\}$ ,  $k = 1, \dots$ . The corresponding BRST vector fields and BRST forms are introduced on  $V\Pi$  as sections of the vector bundles  $\mathcal{V}_E$  and  $\mathcal{V}_E^*$ , respectively. Let us complexify these bundles as  $\mathbf{C} \otimes_{\mathbf{R}} \mathcal{V}_{V\Pi}$  and  $\mathbf{C} \otimes_{\mathbf{R}} \mathcal{V}_{V\Pi}^*$ .

The BRST extension of the form (9.4) on  $V^*Q$  is the 3-form

$$\Omega_S = [d\dot{p}_i \wedge dq^i + dp_i \wedge d\dot{q}^i + i \sum_{k=1}^{\infty} (d\bar{c}_{(k)}^i \wedge dc_{(k)}^i - d\bar{c}_i^{(k)} \wedge d\bar{c}_{(k)}^i)] \wedge dt \quad (9.6)$$

The corresponding bracket of BRST functions on  $V^*Q$  reads

$$\begin{aligned} \{f, g\}_S = \{f, g\}_{V\Pi} + i \sum_{k=1}^{\infty} (-1)^{k[f]} & \left[ \frac{\partial f}{\partial \bar{c}_{(k)}^i} \frac{\partial g}{\partial c_{(k)}^i} + (-1)^k \frac{\partial f}{\partial \bar{c}_{(k)}^i} \frac{\partial g}{\partial c_i^{(k)}} - \right. \\ & \left. \frac{\partial f}{\partial c_i^{(k)}} \frac{\partial g}{\partial \bar{c}_{(k)}^i} - (-1)^k \frac{\partial f}{\partial c_{(k)}^i} \frac{\partial g}{\partial \bar{c}_i^{(k)}} \right]. \end{aligned} \quad (9.7)$$

It satisfies the condition  $\{f, g\}_S = -(-1)^{[f][g]}\{g, f\}_S$ . Then the desired BRST charge takes the form

$$\mathbf{Q} = i[\bar{c}_{(1)}^i (\delta_i^k - a_{ij} \sigma^{jk}) p_k + \sum_{r=1}^{\infty} (\bar{c}_{(2r)}^i a_{ij} \sigma^{jk} c_k^{(2r-1)} + \bar{c}_{(2r+1)}^i (\delta_i^k - a_{ij} \sigma^{jk}) c_k^{(2r)})].$$

Using the bracket (9.7), one can extend this charge in order to obtain the BRST complex for antighosts  $c_i^{(k)}$  and ghosts  $\bar{c}_{(k)}^i$ .

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